Unit #7: Sequence and Series Date: ___________________
Topic: Introduction to Sequences and Series
Objective: SWBAT explain the difference between various sequences and series.

What is an Infinite Sequence?

A *infinite sequence* is ____ generated by a rule.

<u>Formal Definition</u>:

An *infinite sequence* is an ______________________________, usually denoted by _________ , whose domain is the set of ____.

Example #1:

Express the sequence of numbers $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$ as an explicit rule in terms of n.

Does the sequence above converge or diverge?

Example #2:

If $a_n = \left\{\frac{4n}{3+2n}\right\}$, list out the first five terms, then estimate $\lim_{n\to\infty} a_n$.

Practice Problems: Write the first four terms of the sequence and then determine the limit of the sequence or state that the sequence diverges.

1. $a_n = -1 + \frac{(-1)^n}{n}$

2. $b_n = \frac{n^2 - 1}{n}$

3. $a_n = (-1)^{n+1}$

4. $c_n = \frac{n^2}{e^n}$

5. $a_n = \left(1 + \frac{1}{n}\right)^n$

6. $b_n = \frac{3n}{1 - 2n}$

7. $a_n = (-1)^{n+1}\left(1 - \frac{1}{n}\right)$

8. $c_n = \frac{n!}{(n+2)!}$

What is an Infinite Series?

A series is the sum of the terms in a sequence. A finite series has a defined first and last term. An *infinite series* is the sum of an infinite sequence denoted by

$$\sum_{n=1}^{\infty} a_n \quad = \quad a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

In the study of series, the major goal is to determine whether a series has a limit. If the series has a limit, it is said to **converge** to that value. If the series does not have a limit, it is said to **diverge.**

Partial Sums

A **partial sum** is a sum of part of a series. The partial sums of a series form a sequence in themselves.

For an infinite series, we can look at the sequence of partial sums, that is, we can look to see what happens to the sum as we add additional terms.

If the sequence of partial sums converges, then the series converges. Otherwise, we say the series diverges.

Examples: Do the following series converge or diverge?

a) $\sum_{\mathrm{n}=1}^{\infty}(-1)^{\mathrm{n}-1}$

b) $\sum_{\mathrm{n}=1}^{\infty}\frac{1}{n^2}$

c) $\sum_{\mathrm{k}=1}^{\infty}\frac{3}{10^{\mathrm{k}}}$

1. The Geometric Series

In a geometric sequence, each term is obtained by multiplying the preceding term by a constant *r*.

Example:

$-2,\ 1,\ \ -\frac{1}{2},\ \frac{1}{4},\ \ -\frac{1}{8},\ \ldots$ *where* $a = -2$ *is the first term*

and $r = -\frac{1}{2}$ *is the common ratio.*

The geometric series formed by adding successive terms of a geometric sequence is

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

If ________________, then the geometric series ________________________.

If ________________, then the geometric series ________________________.

The sum of the terms of a convergent geometric series is given by $S = \boxed{}$

Some Things to Remember

1. A finite number of terms may be added or subtracted from a series without affecting its convergence or divergence.

   (i.e. $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=m}^{\infty} a_k$ both converge or both diverge)

2. The terms of a series may be multiplied by a non-zero constant without it affecting the convergence or divergence.

3. If $\sum a_n$ and $\sum b_n$ both converge, so does $\sum (a_n + b_n)$.

Practice Problems: Tell whether each series converges or diverges. If it converges, give its sum.

9. $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$

10. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\left(-\frac{1}{2}\right)^{n-1}+\cdots$

11. $\frac{\pi}{2}+\frac{\pi^2}{4}+\frac{\pi^3}{8}+\cdots$

12. $\sum_{k=0}^{\infty}\left(\frac{3}{5}\right)^{k}$

13. $\sum_{n=0}^{\infty}(0.5^n-0.2^n)$

14. $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)\left(\frac{5}{4}\right)^{n}$

15. $3-\frac{9}{2}+\frac{27}{4}-\cdots$

16. $-6-3-\frac{3}{2}-\frac{3}{4}-\cdots$

2. The $p-series$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

If ________________, then the $p - series$ _________________________.

If ________________, then the $p - series$ _________________________.

In the case of the p-series, we can determine **if** the series converges, but **not** the value to which it converges.

Example:

Do each of the following series converge or diverge?

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

b) $\sum_{n=1}^{\infty} \frac{1}{5n^4}$

c) $\sum_{n=1}^{\infty} 2n^{-3}$

3. The Harmonic Series

The *harmonic series* is an example of a p-series where $p = 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

The harmonic series is an important example of a series that __________________, yet has terms that approach zero.

Example:

Does the series $\sum_{k=1}^{\infty} \frac{2}{n}$ converge or diverge?

4. The Telescoping Series

The *telescoping series* is a collapsing series where all the middle terms cancel each other out and the last term always approaches 0, so that all that is left is the **first term**.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) \quad or \quad \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

Example:

Determine whether the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$ converges and if so find its sum.

Practice Problems: Identify each of the following series and then determine whether it converges or diverges.

17. $\sum_{n=0}^{\infty}\left(\frac{3^n}{4^n}\right)$

18. $\sum_{n=0}^{\infty} e^{-n}$

19. $2\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\right]$

20. $\sum_{n=1}^{\infty}\frac{0.5}{n}$

21. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$

22. $\sum_{n=1}^{\infty}\frac{4}{n(n+2)}$

23. $-\frac{1}{7}-\frac{1}{14}-\frac{1}{21}-\frac{1}{28}-\cdots$

24. $\sum_{n=1}^{\infty}\left(\frac{1}{n^3}-0.9^n\right)$

25. $1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots$

26. $\sum_{n=1}^{\infty}\frac{3}{n\sqrt{n}}$

27. $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{n}\right)$

28. $5+\frac{5}{8}+\frac{5}{27}+\frac{5}{64}+\cdots$

29. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n}$

30. $\sum_{n=1}^{\infty}3\left(-\frac{1}{2}\right)^n$

Answers:

1. -1 2. diverges 3. diverges 4. 0 5. 1 6. -3/2 7. diverges

8. 0 9. 6 10. $\frac{2}{3}$ 11. diverges 12. $\frac{5}{2}$ 13. 0.75 14. diverges

15. diverges 16. -12 17. converges 18. converges 19. converges

20. diverges 21. diverges 22. converges 23. diverges 24. converges

25. converges 26. converges 27. diverges 28. converges 29. diverges

30. converges