Unit #8: Taylor Polynomials and Power Series
Topic: Introduction to Taylor Polynomials
Objective: SWBAT develop various Taylor Polynomials for elementary functions.

Warm Up #1:

Find an equation of the tangent line for $f(x) = sinx$ at $x = 0$, then use it to approximate $sin(0.2)$. Is this an over or an under approximation for $sin(0.2)$?

Polynomial functions, such as the tangent line, can be used to approximate other elementary function, such as $sinx$, e^x, lnx, etc. The equation of the tangent line is called a **first-degree Taylor polynomial**.

Taylor polynomials of higher degree can be used to obtain increasingly better approximations of non-polynomial functions within a certain **radius** from a **center of approximation** $x = c$.

Definition of an nth-degree Taylor polynomial:

If f has n derivatives at $x = c$, then the polynomial

$$T_n(x) =$$

is called the nth-degree Taylor polynomial for f centered at c.

There is a specific case of a Taylor polynomial where $c = 0$, such that

$T_n(x) =$

is called the nth-degree Maclaurin polynomial for f.

Note: For Taylor and Maclaurin Polynomials, you must use the $\approx$ symbol so that $f(x) \approx T_n(x)$

Example #1: Find the Maclaurin polynomial of degree $n = 5$ for $f(x) = sinx$. Then use $T_5(x)$ to approximate the value of $sin(0.2)$ using correct notation. Compare your approximation to the actual value of $sin(0.2)$.

Example #2:

Find a third-degree Taylor polynomial centered at 1 for $f(x) = \sqrt{x}$, and use the polynomial to approximate $\sqrt{1.2}$.

Find the error for your approximation and determine the interval in which $\sqrt{1.2}$ could actually live.

Finally, compare your approximation to the actual value of $\sqrt{1.2}$. Is it in your interval?

Practice Problems:

1. Find a Maclaurin polynomial of degree 4 for $f(x) = \frac{1}{1-x}$.

2. Find the Taylor polynomial of degree $n = 6$ for $f(x) = lnx$ at $c = 1$. Then use $T_6(x)$ to approximate the value of ln (1.1).

3. Use the fourth-degree Taylor polynomial for $cosx$ centered at $x = 0$ to approximate $cos\left(\frac{\pi}{6}\right)$.

4. Find the Taylor polynomial of degree $n = 3$ for $f(x) = \frac{1}{x+2}$ at $c = 3$.

5. Find the Taylor polynomial of degree 4 for $f(x) = \sqrt{1-x}$ centered at $c = -3$.

6. Use the fifth-degree Maclaurin polynomial for $f(x) = e^x$.

Warm Up #2:

Suppose that g is a function which has continuous derivatives, and that $g(2) = 3$, $g'(2) = -4$, $g''(2) = 7$, $g'''(2) = -5$. Write he Taylor polynomial of degree 3 for g centered at 2.

Sometimes we can create Maclaurin Polynomials to approximate functions without having to derive them using Taylor's Theorem, but rather by modifying existing polynomials.

Example #3:

List the first four non-zero terms of the Maclaurin Polynomials for $f(x) = sinx$, $f(x) = cosx$, and $f(x) = e^x$, then find the following Maclaurin Polynomials.

a) $g(x) = sin(2x)$

b) $g(x) = xcosx$

c) $g(x) = 4e^{x^2}$

Practice Problems: Find a Taylor polynomial of degree n for each of the following.

7) $f(x) = e^{-x}, n = 3$

8) $f(x) = 2cos(x^2),\ n = 4$

9) $g(x) = ln(x + 2),\ n = 3$

10) $h(x) = \frac{1}{x+1}\ ,\ n = 4$

11) $m(x) = xe^{2x}$, $n = 3$

12) Suppose the function $f(x)$ is approximated near $x = 0$ by a sixth-degree Taylor polynomial $P_6(x) = 3x - 4x^3 + 5x^6$. Give the value of each of the following:

(a) $f(0)$ (b) $f'(0)$ (c) $f'''(0)$

(d) $f^5(0)$ (e) $f^6(0)$

13) **Calculator Allowed**

Suppose that g is a function which has continuous derivatives, and that $g(5) = 3$, $g'(5) = -2,\ g''(5) = 1,\ g'''(5) = -3$.

(a) What is the Taylor polynomial of degree 2 for g near 5. What is the Taylor polynomial of degree 3 near 5?

(b) Use the two polynomials that you found in part (a) to approximate $g(4.9)$.

Answer Key

1) $T_4(x) = 1 + x + x^2 + x^3 + x^4$	2) $T_6(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6}$
3) $T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$	4) $T_3(x) = \frac{1}{5} - \frac{(x-3)}{25} + \frac{(x-3)^2}{125} - \frac{(x-3)^3}{625}$
5) $T_4(x) = 1 - \frac{(x+3)}{4} - \frac{(x+3)^2}{64} - \frac{(x+3)^3}{512} - \frac{5(x+3)^4}{16384}$	6) $T_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$
7) $T_3(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$	8) $T_4(x) = 2 - x^4 + \frac{x^8}{12}$
9) $T_3(x) = (x+1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3}$	10) $T_4(x) = 1 - x + x^2 - x^3 + x^4$
11) $T_3(x) = x + 2x^2 + 2x^3 + \frac{4x^4}{3}$	12) a) 0 b) 3 c) -24 d) 0 e) 3600
13) a) $T_2(x) = 3 - 2(x-5) + \frac{(x-5)^2}{2}$; $T_3(x) = 3 - 2(x-5) + \frac{(x-5)^2}{2} - \frac{(x-5)^3}{2}$ b) $T_2(4.9) = 2.805,\quad T_3(4.9) = 2.8055$	