

Unit #5: Improper Integrals

Topic: One-Sided Improper Integrals

Objective: SWBAT integrate improper integrals by using limits.

Warm Up #3:

Evaluate each of the following using your calculator:

1) $\int_1^{100} \frac{1}{x} dx$

2) $\int_1^{1000} \frac{1}{x} dx$

3) $\int_1^{1,000,000} \frac{1}{x} dx$

Based on the values above what do you think $\int_1^{\infty} \frac{1}{x} dx$ equals?

4) $\int_1^{100} \frac{1}{x^2} dx$

5) $\int_1^{1000} \frac{1}{x^2} dx$

6) $\int_1^{10,000} \frac{1}{x^2} dx$

Based on the values above what do you think $\int_1^{\infty} \frac{1}{x^2} dx$ equals?**Improper Integrals**

There are two cases in which integrals are called improper integrals.

Case 1: One or both of the limits of the integral are ∞ or $-\infty$.

$$\int_a^{+\infty} f(x) dx, \int_{-\infty}^b f(x) dx, \text{ or } \int_{-\infty}^{+\infty} f(x) dx.$$

They are evaluated by rewriting the integral as a proper integral and then using limits.

1. If $f(x)$ is continuous in the interval $[a, \infty)$, then: $\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$ 2. If $f(x)$ is continuous in the interval $(-\infty, b]$, then:

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

If the limit exists, the integral is said to *converge* to that number. If no limit exists, the integral is said to *diverge*.

Example #1:

Find the value of $\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx$.

Example #2:

Find the value of $\int_{-\infty}^0 e^{-x} dx$

Problem Set #3: Evaluate each of the following integrals and determine whether or not they converge or diverge.

1) $\int_4^{\infty} \frac{1}{\sqrt{x}} dx$

2) $\int_{-\infty}^0 2e^{8x} dx$

$$3) \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$4) \int_0^{\infty} \frac{e^x}{1+e^x} dx$$

$$5) \int_{-\infty}^2 \frac{2}{x^2+4} dx$$

$$6) \int_1^{\infty} \frac{1}{x^4} dx$$

$$7) \int_1^{\infty} \frac{2+x}{x^2} dx$$

$$8) \int_{-\infty}^1 \frac{dx}{(x-2)^2}$$

9) $\int_0^{\infty} \frac{x}{x^2 + 1} dx$

10) $\int_{-\infty}^0 e^{5x} dx$

11) $\int_2^{\infty} \frac{dx}{x(\ln x)^2}$

12) $\int_{-1}^{\infty} \frac{dx}{x^2 + 5x + 6}$

13) $\int_{-\infty}^0 \frac{1}{(x-2)^3} dx$

14) $\int_1^{\infty} \frac{1}{\sqrt[4]{x}} dx$

Warm Up #4:

Find $\lim_{x \rightarrow 0} \frac{1 - \cos^2(2x)}{x^2}$

Sometimes an integral can be doubly improper.



3. If $f(x)$ is continuous in the interval $(-\infty, \infty)$, then:

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx + \lim_{c \rightarrow \infty} \int_b^c f(x) dx$$

where b is any number.

Note as well that this requires **BOTH** of the integrals to be convergent in order for this integral to also be convergent.

If **either** of the two integrals is **divergent** then so is this integral.

Example #3:

Find the value of $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

Problem Set #4: Evaluate each of the following integrals and determine whether or not they converge or diverge.

$$15) \int_{-\infty}^{\infty} e^{2x} dx$$

$$16) \int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2} dx$$

$$17) \int_{-\infty}^{\infty} 4xe^{-5x^2} dx$$

$$18) \int_{-\infty}^{\infty} \frac{6x^3}{(x^4+1)^2} dx$$

19) $\int_{-\infty}^{\infty} (2 - x^4) dx$

20) $\int_{-\infty}^{\infty} e^{-\frac{x}{4}} dx$

21) $\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx$

22) $\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx$

(Hint: Think of
denominator as
 $1 + (x^2)^2$)

Answers:

1. D 2. $\frac{1}{4}$ 3. $\frac{\pi}{2}$ 4. D 5. $\frac{3\pi}{4}$ 6. $\frac{1}{3}$ 7. D 8. 1 9. D 10. $\frac{1}{5}$ 11. $\frac{1}{\ln 2}$ 12. $\ln 2$
13. $-\frac{1}{8}$ 14. D 15. D 16. 0 17. 0 18. 0 19. D 20. D 21. D 22. 0

Name _____ Date _____

Homework - More Practice with Improper Integrals*Directions: Evaluate each of the following integrals.*

1) $\int_e^{\infty} \frac{1}{x(\ln x)^3} dx$

Answer: 1/2

2) $\int_{-\infty}^{-1} \frac{1}{\sqrt{2-x}} dx$

Answer: Diverges

3) $\int_0^{\infty} \frac{16 \arctan x}{1+x^2} dx$

Answer: $2\pi^2$

4) $\int_{-\infty}^0 e^{x+2} dx$

Answer: e^2

5) $\int_{-1}^{\infty} \frac{3}{(3x+5)^4} dx$

Answer: 1/24

6) $\int_0^{\infty} \cos x dx$

Answer: -1

7) $\int_1^{\infty} \frac{1}{\sqrt[4]{x^3}} dx$

Answer: diverges

8) $\int_{-\infty}^{\infty} x e^{-x^2} dx$

Answer: 0

