

Unit #4: Area and Volume

Topic: Finding the Volume of Solids with Known Cross Sections

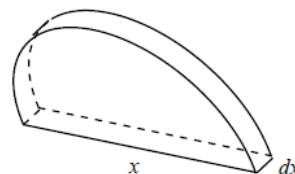
Objective: *SWBAT find the volume of a solid with a known cross section.*

## Warm Up #4:

Find the area of the region bounded by the graphs of  $y^2 = 3 - x$  and  $y = x - 1$ .

We used definite integrals to find the area of a given region by slicing the region and adding up the areas of the slices.

We will use definite integrals to compute volume in a similar way, by slicing the solid and adding up the volumes of the slices.



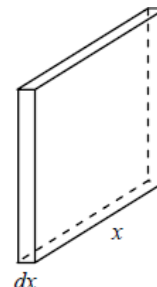
### ***What is a cross section?***

Imagine a loaf of bread. Now imagine the shape of a slice through the loaf of bread. This shape would be a cross section. Technically a *cross section* of a three dimensional figure is the intersection of a plane and that figure. It would be like cutting an object and then looking at the face of where you just cut.

The cross sections we will be dealing are almost entirely perpendicular to the  $x - axis$ .

## To find the volume:

1. Sketch the solid and a typical cross section.
2. Find a formula for  $A(x)$ .
3. Find the limits of integration.
4. Integrate  $A(x)$  to find the volume.



## So...

The volume of a solid of known integrable cross section area  $A(x)$  from  $x = a$  to  $x = b$  is the integral of  $A$  from  $a$  to  $b$

$$V = \int_a^b A(x) dx$$

### Example #1:

Imagine that the floor of a building is the region bounded by  $x = y^2$  and the line  $x = 9$ .

A tent or building is formed by placing vertical cross-sections along this base where each cross-section is a square of perfect size to cross the parabolic region on the floor.

Each square cross-section is perpendicular to the x-axis, so the thickness of the cross-section is represented by  $dx$ . Find the volume.

What is the volume if the cross sections are semi-circles?

What is the volume if the cross sections are equilateral triangles?

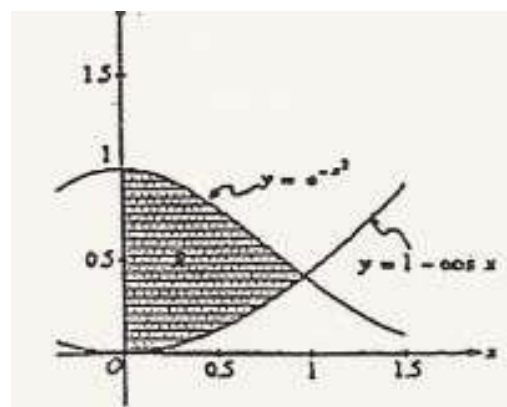
*Problem Set #4:*

- 1) Find the volume if the base of a solid is the region bounded by  $x = 1$  and  $x = 2 - y^2$  and cross sections are squares perpendicular to the  $x - axis$ .

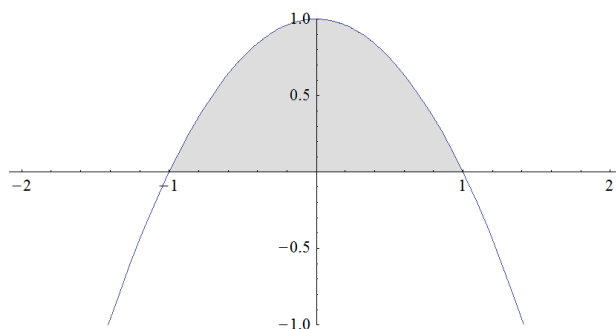
- 2) The base of a solid is a region in the first quadrant bounded by the  $x$ -axis, the  $y$ -axis, and the line  $x + 2y = 8$ . If the cross sections of the solid perpendicular to the  $x$  - *axis* are semicircles, what is the volume of the solid?
- 3) The base of a solid is the region enclosed by the graph of  $y = e^{-x}$ , the coordinate axes, and the line  $x = 3$ . If all cross sections perpendicular to the  $x$  - *axis* are squares, what is its volume?

- 4) Find the volume of the solid whose base is bounded by the graphs of  $y = x + 1$  and  $y = x^2 + 1$ , where the cross sections taken perpendicular to the  $x$ -axis are equilateral triangles.

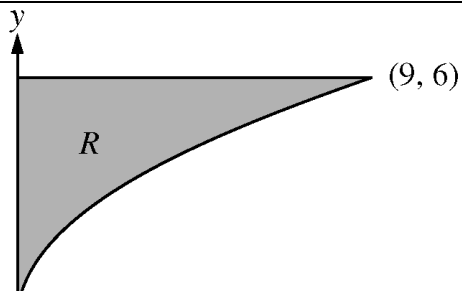
- 5) The base of a solid is the region in the first quadrant bounded by the graphs of  $y = e^{-x^2}$ ,  $y = 1 - \cos x$ , and the  $y$ -axis. For this solid, each cross section perpendicular to the  $x$ -axis is a square. Find the volume of the solid.



- 6) The base of a solid is the region bounded by the graph of  $y = -x^2 + 1$  and the  $x$ -axis. For this solid, each cross section perpendicular to the  $x$ -axis is a rectangle with height three times the base. What is the volume of this solid?



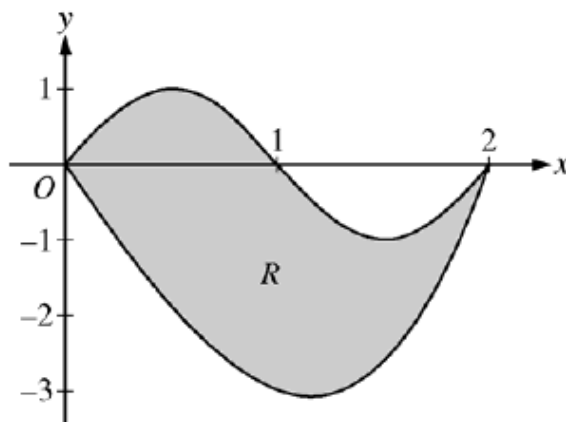
7) 2010 AB #4



Let  $R$  be the region in the first quadrant bounded by the graph of  $y = 2\sqrt{x}$ , the horizontal line  $y = 6$ , and the  $y$ -axis, as shown in the figure above.

- (a) Find the area of  $R$
- (b) Region  $R$  is the base of a solid. For each  $y$ , where  $0 \leq y \leq 6$ , the cross section of the solid taken perpendicular to the  $y$ -axis is a rectangle whose height is 3 times the length of its base in region  $R$ . Write, but do not evaluate, an integral expression that gives the volume of the solid.

8) 2008 BC #1



Let  $R$  be the region bounded by the graphs of  $y = \sin(\pi x)$  and  $y = x^3 - 4x$ , as shown in the figure above.

- Find the area of  $R$ .
- The horizontal line  $y = -2$  splits the region  $R$  into two parts. Write, but do not evaluate, an integral expression for the area of the part of  $R$  that is below this horizontal line.
- The region  $R$  is the base of a solid. For this solid, each cross section perpendicular to the  $x$ -axis is a square. Find the volume of this solid.
- The region  $R$  models the surface of a small pond. At all points in  $R$  at a distance  $x$  from the  $y$ -axis, the depth of the water is given by  $h(x) = 3 - x$ . Find the volume of water in the pond.

9) 2007 BC #1

Let  $R$  be the region in the first and second quadrants bounded above by the graph of  $y = \frac{20}{1+x^2}$  and below by the horizontal line  $y = 2$ .

- (a) Find the area of  $R$ .
- (b) The region  $R$  is the base of a solid. For this solid, the cross sections perpendicular to the  $x$ -axis are semicircles. Find the volume of this solid.



10) Let  $R$  be the region bounded by the graphs of  $y = \sin(x^2)$  and  $y = 1 - x^2$ .

(a) Find the area of  $R$ .

(b) The region  $R$  is the base of a solid. For this solid, each cross section perpendicular to the  $x$ -axis is an isosceles right triangle with one leg in the region  $R$ . Find the volume of this solid.

Answer Key:

1. 2	2. 16.755	3. $-\frac{1}{2e^6} + \frac{1}{2}$	4. .0144
5. 0.461	6. 3.2	7. a) 18	b) $V = \frac{3}{16} \int_0^6 y^4 dy$
8. a) 4	b) $V = \int_{.53919}^{1.6751} -2 - (x^3 - 4x) dx$	c) 9.978	d) 8.370
9. a) 37.962	b) 174.268	10. a) .947	b) .474

