Unit #6: Parametric and Polar Derivatives
Topic: Planar Motion
Objective: SWBAT solve free response problems involving planar motion.

Warm Up #4:

2013 AB #2 **CALCULATOR**

A particle moves along a straight line. For $0 \le t \le 5$, the velocity of the particle is given by $v(t) = -2 + \left(t^2 + 3t\right)^{6/5} - t^3$, and the position of the particle is given by $s(t)$. It is known that $s(0) = 10$.

(a) Find all values of t in the interval $2 \le t \le 4$ for which the speed of the particle is 2.

(b) Write an expression involving an integral that gives the position $s(t)$. Use this expression to find the position of the particle at time $t = 5$.

(c) Find all times t in the interval $0 \le t \le 5$ at which the particle changes direction. Justify your answer.

(d) Is the speed of the particle increasing or decreasing at time $t = 4$? Give a reason for your answer.

Planar Motion

When an object moves along a straight line, its velocity and direction can be determined by a single number. Vectors can be used to model the motion of objects moving in a coordinate plane.

Using what we already know about relating position, velocity, and acceleration of an object moving along a straight line we can now make similar relations for an object moving in the coordinate plane.

Some definitions to know:

Velocity, Speed, Acceleration, and Direction of Motion

Suppose a particle moves along a smooth curve in the plane so that its position at any time t is $(x(t)), y(t)$, where x and y are differentiable functions of t.

1. Position vector is $s(t) = \langle x(t), y(t) \rangle$
2. Velocity vector is $v(t) = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$
3. Speed is the magnitude of v, denoted $|v| = \sqrt{(v_1(t))^2 + (v_2(t))^2}$
   Speed is a *scalar*, **not** a vector.
4. Acceleration vector is $a(t) = \langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \rangle$

Displacement and Distance Traveled

Suppose a particle moves along a path in the plane so that its velocity at any time t is $v(t) = \langle v_1(t), v_2(t) \rangle$, where v_1 and v_2 are integrable functions of t.

The displacement from $t = a \text{ to } t = b$ is $\langle \int_a^b \boldsymbol{v_1(t)dt}, \int_a^b \boldsymbol{v_2(t)dt} \rangle$
The vector is then added to the position at time $t = a$ to get the position at time $t = b$.

Distance traveled from $t = a \text{ to } t = b$ is $\int_a^b \boldsymbol{|v(t)|dt} = \int_a^b \sqrt{\boldsymbol{(v_1(t))^2 + (v_2(t))^2}}\, \boldsymbol{dt}$

1. *2008 BC #1 Form B* ***(CALCULATOR)***

A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = \sqrt{3t} \text{ and } \frac{dy}{dt} = 3\cos\left(\frac{t^2}{2}\right).$$

The particle is at position $(1, 5)$ at time $t = 4$.

(a) Find the acceleration vector at time $t = 4$.

(b) Find the y-coordinate of the position of the particle at time $t = 0$.

(c) On the interval $0 \leq t \leq 4$, at what time does the speed of the particle first reach 3.5 ?

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 4$.

2. *2007 BC #2 Form B* ***(CALCULATOR)***

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \arctan\left(\frac{t}{1+t}\right) \text{ and } \frac{dy}{dt} = \ln\left(t^2 + 1\right)$$

for $t \geq 0$. At time $t = 0$, the object is at position $(-3, -4)$. (Note: $\tan^{-1} x = \arctan x$)

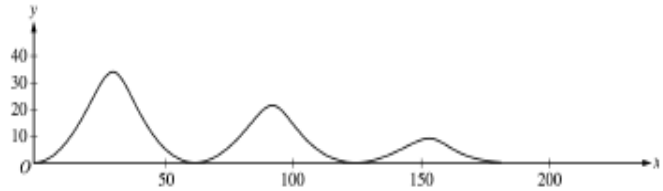
(a) Find the speed of the object at time $t = 4$.

(b) Find the total distance traveled by the object over the time interval $0 \leq t \leq 4$.

(c) Find $x(4)$.

(d) For $t > 0$, there is a point on the curve where the line tangent to the curve has slope 2. At what time t is the object at this point? Find the acceleration vector at this point.

3. *2002 BC #3* ***(CALCULATOR)***



The figure above shows the path traveled by a roller coaster car over the time interval $0 \le t \le 18$ seconds. The position of the car at time t seconds can be modeled parametrically by $x(t) = 10t + 4\sin t$, $y(t) = (20 - t)(1 - \cos t)$, where x and y are measured in meters. The derivatives of these functions are given by

$x'(t) = 10 + 4\cos t$, $y'(t) = (20 - t)\sin t + \cos t - 1$.

(a) Find the slope of the path at time $t = 2$. Show the computations that lead to your answer.

(b) Find the acceleration vector of the car at the time when the car's horizontal position is $x = 140$.

(c) Find the time t at which the car is at its maximum height, and find the speed, in m/sec, of the car at this time.

(d) For $0 < t < 18$, there are two times at which the car is at ground level ($y = 0$). Find these two times and write an expression that gives the average speed, in m/sec, of the car between these two times. Do not evaluate the expression.

4. *2005 BC #1 Form B* ***(CALCULATOR)***

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = 12t - 3t^2 \text{ and } \frac{dy}{dt} = \ln\left(1 + (t-4)^4\right).$$

At time $t = 0$, the object is at position $(-13, 5)$. At time $t = 2$, the object is at point P with x-coordinate 3.

(a) Find the acceleration vector at time $t = 2$ and the speed at time $t = 2$.

(b) Find the y-coordinate of P.

(c) Write an equation for the line tangent to the curve at P.

(d) For what value of t, if any, is the object at rest? Explain your reasoning.

5. *1993 BC #2* ***(CALCULATOR)***

The position of a particle at any time $t \geq 0$ is given by $x(t) = t^2 - 3$ and $y(t) = \frac{2}{3}t^3$.

(a) Find the magnitude of the velocity vector at $t = 5$.

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

(c) Find $\frac{dy}{dx}$ as a function of x.

6. *2000 BC #4* ***(NO CALCULATOR)***

A moving particle has position $(x(t), y(t))$ at time t. The position of the particle at time $t = 1$ is (2,6) and the velocity vector at any time $t > 0$ is given by $\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)$.

(a) Find the acceleration vector at time $t = 3$.

(b) Find the position of the particle at time $t = 3$.

(c) For what time $t > 0$ does the line tangent to the path of the particle at $(x(t), y(t))$ have a slope of 8?

(d) The particle approaches a line as $t \to \infty$. Find the slope of this line. Show the work that leads to your conclusion.

7. *2006 BC #2 Form B* ***(CALCULATOR)***

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \tan\left(e^{-t}\right) \text{ and } \frac{dy}{dt} = \sec\left(e^{-t}\right)$$

for $t \geq 0$. At time $t = 1$, the object is at position $(2, -3)$.

(a) Write an equation for the line tangent to the curve at position $(2, -3)$.

(b) Find the acceleration vector and the speed of the object at time $t = 1$.

(c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 2$.

(d) Is there a time $t \geq 0$ at which the object is on the y-axis? Explain why or why not.

8. *2010 BC #2 Form B* ***(CALCULATOR)***

The velocity vector of a particle moving in the plane has components given by

$$\frac{dx}{dt} = 14\cos\left(t^2\right)\sin\left(e^t\right) \text{ and } \frac{dy}{dt} = 1 + 2\sin\left(t^2\right), \text{ for } 0 \le t \le 1.5.$$

At time $t = 0$, the position of the particle is $(-2, 3)$.

(a) For $0 < t < 1.5$, find all values of t at which the line tangent to the path of the particle is vertical.

(b) Write an equation for the line tangent to the path of the particle at $t = 1$.

(c) Find the speed of the particle at $t = 1$.

(d) Find the acceleration vector of the particle at $t = 1$.

9. *2010 BC #3* ***(CALCULATOR)***

A particle is moving along a curve so that its position at time t is $(x(t), y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ is not explicitly given. Both x and y are measured in meters, and t is measured in seconds. It is known that $\frac{dy}{dt} = te^{t-3} - 1$.

(a) Find the speed of the particle at time $t = 3$ seconds.

(b) Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

(c) Find the time t, $0 \le t \le 4$, when the line tangent to the path of the particle is horizontal. Is the direction of motion of the particle toward the left or toward the right at that time? Give a reason for your answer.

(d) There is a point with x-coordinate 5 through which the particle passes twice. Find each of the following.

(i) The two values of t when that occurs

(ii) The slopes of the lines tangent to the particle's path at that point

(iii) The y-coordinate of that point, given $y(2) = 3 + \frac{1}{e}$

10. *2003 #4 Form B* ***(NO CALCULATOR)***

A particle moves in the xy-plane so that the position of the particle at any time t is given by

$$x(t) = 2e^{3t} + e^{-7t} \quad \text{and} \quad y(t) = 3e^{3t} - e^{-2t}.$$

(a) Find the velocity vector for the particle in terms of t, and find the speed of the particle at time $t = 0$.

(b) Find $\frac{dy}{dx}$ in terms of t, and find $\lim_{t \to \infty} \frac{dy}{dx}$.

(c) Find each value t at which the line tangent to the path of the particle is horizontal, or explain why none exists.

(d) Find each value t at which the line tangent to the path of the particle is vertical, or explain why none exists.

11. *2011 BC #1* ***(CALCULATOR)***

At time t, a particle moving in the xy-plane is at position $(x(t), y(t))$, where $x(t)$ and $y(t)$ are not explicitly given. For $t \geq 0$, $\frac{dx}{dt} = 4t + 1$ and $\frac{dy}{dt} = \sin\left(t^2\right)$. At time $t = 0$, $x(0) = 0$ and $y(0) = -4$.

(a) Find the speed of the particle at time $t = 3$, and find the acceleration vector of the particle at time $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at time $t = 3$.

(c) Find the position of the particle at time $t = 3$.

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

12. *2012 BC #2* ***(CALCULATOR)***

For $t \geq 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 2$, the particle is at position $(1, 5)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+2}}{e^t}$ and $\frac{dy}{dt} = \sin^2 t$.

(a) Is the horizontal movement of the particle to the left or to the right at time $t = 2$? Explain your answer. Find the slope of the path of the particle at time $t = 2$.

(b) Find the x-coordinate of the particle's position at time $t = 4$.

(c) Find the speed of the particle at time $t = 4$. Find the acceleration vector of the particle at time $t = 4$.

(d) Find the distance traveled by the particle from time $t = 2$ to $t = 4$.

Answer Key

1)a) $\langle .433, -11.872\rangle$ b) 1.601 c) 2.226 d) 13.182

2)a) 2.912 b) 6.423 c) -0.892 d) t =1.358 ; $\langle 0.135, 0.955\rangle$

3)a) 1.794 b) $\langle -3.529, 1.226\rangle$ c) $t = 3.024$; 6.028 d) $t = 2\pi, 4\pi$

$$avg\ speed = \frac{1}{4\pi - 2\pi}\int_{2\pi}^{4\pi}\sqrt{(10+4cost)^2 + ((20-t)sint + cost - 1)^2}\,dt$$

4)a) $\langle 0, -1.882\rangle$; 12.330 b) 13.671 c) $y - 13.671 = \frac{ln17}{12}(x-3)$ d) $t = 4$

5)a) 50.990 b) 87.716 c) $t = \sqrt{x+3}$

6)a) $\langle \frac{2}{27}, -\frac{2}{27}\rangle$ b) $\left(\frac{10}{3}, \frac{32}{3}\right)$ c) $t = \sqrt{\frac{3}{2}}$ d) 2

7)a) $y + 3 = 2.781(x-2)$ b) $\langle -0.423, -0.152\rangle$; 1.139 c) 1.059 d) no

8)a) $t = 1.145\ \ and\ \ t = 1.253$ b) $y - 4.621 = 0.863(x - 9.315)$ c) 4.105
d) $\langle -28.425, 2.161\rangle$

9)a) 2.828 b) 11.588 c) $t = 2.208$ d) $t = 1\ and\ t = 3$; .432 and 1 ; 4

10)a) $\langle 6e^{3t} - 7e^{-7t}, 9e^{3t} + 2e^{-2t}\rangle$; $\sqrt{122}$ b) 3/2 c) none d) $t = \frac{1}{10}ln\frac{7}{6}$

11)a) 13.007 ; $\langle 4, .-5.466\rangle$ b) 0.032 c) (21,-3.226) d) 21.091

12)a) right; 3.055 b) 1.253 c) 0.575 ; $\langle -0.041, 0.989\rangle$ d) 0.651