Unit \#8: Taylor Polynomials and Power Series
Topic: Taylor and Maclaurin Series
Objective: SWBAT manipulate various Taylor and Maclaurin series.

## Warin Up 带馬:

Let $P(x)=3 x^{2}-5 x^{3}+7 x^{4}+3 x^{5}$ be the fifth-degree Taylor polynomial for the function $f$ about $x=0$. What is the value of $f^{\prime \prime \prime}(0)$ ?
(a) -30
(b) -15
(c) -5
(d) $-\frac{5}{6}$
(e) $-\frac{1}{6}$

## Differentiation \& Integration of Power Series

The derivative and the antiderivative of a power series are again power series.
The derivative and antiderivative of a power series are found by integrating and differentiating each individual term in the original series.

Example \#1:
a) Derive a Maclaurin series for $\sin x$, and then take its derivative.

What do you notice about the new series you end up with??
b) If $f(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots \frac{x^{n}}{n!}+\cdots$, and $F(x)=\int f(x) d x$ and $F(0)=1$, find $F(x)$.

The intervals of convergence of the derivative and antiderivative of a power series have the same center and radius as the interval of convergence of the original power series, BUT the endpoints may differ.

The interval of convergence of the derivative of a power series may lose endpoints, while the antiderivative of a power series may gain endpoints.

Example \#2:
a) Write the power series for $\ln x$ centered at $x=1$, and its interval of convergence.
b) Find the power series for the derivative of $\ln x$ and for the antiderivative of $\ln x$. State the interval of convergence for each power series.

## Practice Problems:

1) Let $f$ be a function that has derivatives of all orders for all real numbers. Assume $f(1)=3, f^{\prime}(1)=-2, f^{\prime \prime}(1)=2, f^{\prime \prime \prime}(1)=4$.
a) Write the second-degree Taylor polynomial for $f$ about $x=1$ and use it to approximate $f(0.7)$.
b) Write the third-degree Taylor polynomial for $f$ about $x=1$ and use it to approximate $f(1.2)$.
c) Write the second-degree Taylor polynomial for $f^{\prime}$, the derivative of $f$, about $x=1$ and use it to approximate $f^{\prime}(1.2)$.
2) Let $P(x)=7-3(x-4)+5(x-4)^{2}-2(x-4)^{3}+6(x-4)^{4}$ be the 4 -degree Taylor polynomial for the function $f$ about 4. Assume $f$ has derivatives for all orders for all real numbers.
a) Find $f(4)$ and $f^{\prime \prime \prime}(4)$.
b) Write the second-degree Taylor polynomial for $f^{\prime}$ about 4 and use it to approximate $f^{\prime}(4.3)$.
c) Write the fourth-degree Taylor polynomial for $g(x)=\int_{4}^{x} f(t) d t$ about 4 .
3) The Taylor series about $x=5$ for a certain function $f$ converges to $f(x)$ for all $x$ in the interval of convergence. The $n$th derivative of $f$ at $x=5$ is given by $f^{(n)}(5)=\frac{(-1)^{n} n!}{2^{n}(n+2)}$, and $f(5)=\frac{1}{2}$.
(a) Write the third-degree Taylor polynomial for $f$ about $x=5$.
(b) Find the radius of convergence of the Taylor series for $f$ about $x=5$.
(c) Show that the sixth-degree Taylor polynomial for $f$ about $x=5$ approximates $f(6)$ with error less than $\frac{1}{1000}$.
4) The Maclaurin series for the function $f$ is given by

$$
f(x)=\sum_{n=0}^{\infty} \frac{(2 x)^{n+1}}{n+1}=2 x+\frac{4 x^{2}}{2}+\frac{8 x^{3}}{3}+\frac{16 x^{4}}{4}+\cdots+\frac{(2 x)^{n+1}}{n+1}+\cdots
$$

on its interval of convergence.
(a) Find the interval of convergence of the Maclaurin series for $f$. Justify your answer.
(b) Find the first four terms and the general term for the Maclaurin series for $f^{\prime}(x)$.
(c) Use the Maclaurin series you found in part (b) to find the value of $f^{\prime}\left(-\frac{1}{3}\right)$.
5) A function $f$ is defined by

$$
f(x)=\frac{1}{3}+\frac{2}{3^{2}} x+\frac{3}{3^{3}} x^{2}+\cdots+\frac{n+1}{3^{n+1}} x^{n}+\cdots
$$

for all $x$ in the interval of convergence of the given power series.
(a) Find the interval of convergence for this power series. Show the work that leads to your answer.
(b) Find $\lim _{x \rightarrow 0} \frac{f(x)-\frac{1}{3}}{x}$.
(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_{0}^{1} f(x) d x$.
(d) Find the sum of the series determined in part (c).
6) Let $f(x)=\ln \left(1+x^{3}\right)$.
(a) The Maclaurin series for $\ln (1+x)$ is $x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{4}+\cdots+(-1)^{n+1} \cdot \frac{x^{n}}{n}+\cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for $f$.
(b) The radius of convergence of the Maclaurin series for $f$ is 1 . Determine the interval of convergence. Show the work that leads to your answer.
(c) Write the first four nonzero terms of the Maclaurin series for $f^{\prime}\left(t^{2}\right)$. If $g(x)=\int_{0}^{x} f^{\prime}\left(t^{2}\right) d t$, use the first two nonzero terms of the Maclaurin series for $g$ to approximate $g(1)$.
(d) The Maclaurin series for $g$, evaluated at $x=1$, is a convergent alternating series with individual terms that decrease in absolute value to 0 . Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.
7) Let $f$ be the function given by $f(x)=e^{-x^{2}}$.
a) Write the first four nonzero terms and the general term of the Taylor series for $f$ about $x=0$.
b) Use your answer to part (a) to find $\lim _{x \rightarrow 0} \frac{1-x^{2}-f(x)}{x^{4}}$.
c) Write the first four nonzero terms of the Taylor series for $\int_{0}^{x} e^{-t^{2}} d t$ about $x=0$. Use the first two terms of your answer to estimate $\int_{0}^{1 / 2} e^{-t^{2}} d t$.
d) Explain why the estimate found in part (c) differs from the actual value of $\int_{0}^{1 / 2} e^{-t^{2}} d t$ by less than $\frac{1}{200}$.

## Answer Key:



