

Unit #3: Differential Equations

Topic: Exponential Growth and Decay

Objective: *SWBAT solve exponential growth and decay problems by using differential equations.*

## Warm Up #7:

Suppose that a population of fruit flies grows exponentially. If there were 100 flies after the second day and 300 flies after the fourth day, how many flies were in the original population?

In many applications, the rate of change of a variable  $y$  is proportional to the value of  $y$ . If  $y$  is a function of time  $t$ , then the differential equation

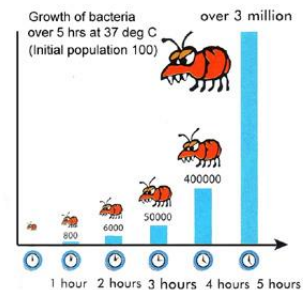


where  $k$  is the **growth constant** (if positive) or the **decay constant** (if negative), describes this relationship.

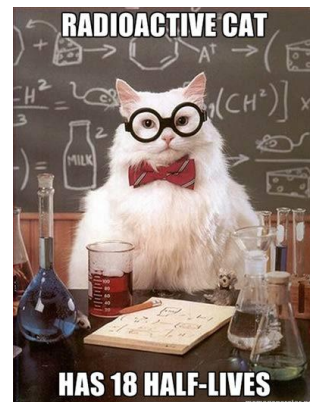
Let's solve this differential equation and see what happens...

*Example #1: 1988 AP Calculus BC*

Bacteria in a certain culture increases at a rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

*Example #2:*

Radium-226 decays at a rate proportional to the quantity present. Its half-life is 1612 years. How long will it take for one quarter of a given quantity of radium-226 to decay?



*Problem Set #7:*

- 1) The number of bacteria in a culture is growing at a rate of  $3000e^{2t/5}$  per unit of time  $t$ . At  $t = 0$ , the number of bacteria present was 7,500. Find the number present at  $t = 5$ .
- 2) The rate at which a radioactive element decays (as measured by the number of nuclei that change per unit of time) is approximately proportional to the amount of nuclei present. Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram if Pu-239 has a half-life of 24,360 years?
- 3) During a certain epidemic, the number of people that are infected at any time increases at a rate proportional to the number of people that are infected at that time. If 1,000 people are infected when the epidemic is first discovered, and 1,200 are infected 7 days later, how many people are infected 12 days after the epidemic is first discovered?
- 4) A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, how much will the puppy weigh when its 3 months old?

- 5) The half-life  $T$  of radium is 1690 years. How much will remain of one gram of radium after 1000 years?
- 6) Population  $y$  grows according to the equation  $\frac{dy}{dt} = ky$ , where  $k$  is a constant and  $t$  is measured in years. If the population doubles every 10 years, then find the value of  $k$ .
- 7) Assume the number of bacteria in a culture grows exponentially with a growth constant of 0.02 where time is measured in hours.
- (a) How much bacteria will be present after one hour if the initial number was 1,000?
  - (b) Given the same initial 1000 bacteria, in how many hours will there be 100,000 bacteria?
- 8) Scientists who use carbon-14 dating use 5700 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present has decayed.

## Newton's Law of Cooling

According to Newton's law of cooling, a hot object cools at a rate proportional to the difference between its own temperature and that of its environment.



*For Example:* Espresso left in a cup will cool to the temperature of the surrounding air. The rate of cooling is proportional to the difference in temperature between the liquid and the air. (It is assumed that the air temperature is constant.)

If we solve the differential equation:  $\frac{dT}{dt} = -k[T - T_s]$  we get

### Newton's Law of Cooling

$$T - T_s = [T_0 - T_s] e^{-kt}$$

where  $T_s$  is the temperature of the surrounding medium, which is a constant.

### Example #3:

A hard-boiled egg at  $98^\circ\text{C}$  is put in a pan under running  $18^\circ\text{C}$  water to cool. After 5 minutes, the egg's temperature is found to be  $38^\circ\text{C}$ . How much longer will it take the egg to reach  $20^\circ\text{C}$ ?

*Problem Set #8:*

- 9) If a roast at room temperature,  $68^{\circ}\text{F}$ , is put into a  $20^{\circ}\text{F}$  freezer, and if, after 2 hours the temperature of the roast is  $40^{\circ}\text{F}$
- What is the temperature of the roast after 5 hours?
  - How long will it take for the temperature of the roast to fall to  $21^{\circ}\text{F}$ ?
- 10) A thermometer is taken from a room that is  $20^{\circ}\text{C}$  to the outdoors where the temperature is  $5^{\circ}\text{C}$ . After one minute, the thermometer reads  $12^{\circ}\text{C}$ . Use Newton's Law of Cooling to answer the following questions.
- What will the reading on the thermometer be after one more minute?
  - When will the thermometer read  $6^{\circ}\text{C}$ ?

- 11) Suppose that a cup of soup cooled from  $90^{\circ}\text{C}$  to  $60^{\circ}\text{C}$  in 10 minutes in a room whose temperature was  $20^{\circ}\text{C}$ . Use Newton's Law of Cooling to answer the following questions.
- How much longer would it take the soup to cool to  $35^{\circ}\text{C}$ ?
  - Instead of being left to stand in the room, the cup of  $90^{\circ}\text{C}$  soup is put into a freezer whose temperature is  $-15^{\circ}\text{C}$ . How long will it take the soup to cool from  $90^{\circ}\text{C}$  to  $35^{\circ}\text{C}$ ?
- 12) In a murder investigation, a corpse was found by a detective at exactly 8 P.M. Being alert, the detective also measured the body temperature and found it to be  $70^{\circ}\text{F}$ . Two hours later, the detective measured the body temperature again and found it to be  $60^{\circ}\text{F}$ . If the room temperature is  $50^{\circ}\text{F}$ , and assuming that the body temperature of the person before death was  $98.6^{\circ}\text{F}$ , at what time did the murder occur?

- 13) After spending many hours study for your math class you decide to reward yourself with a case of Stiffler Soda. Unfortunately I.V. Market only had warm case left, the physics students got there first and got all of the cold ones. You buy several bags of ice and decide to chill it at home. When you get home the temperature of the soda is 76 degrees F and you proceed to pack it in ice. After ten minutes the temperature has cooled to 70 degrees F. If Stiffler Soda is best served at 60 degrees F or cooler, how much longer must you wait to enjoy your Sriffler Soda?
- 14) An extra large double latte frappuccino au lait is handed to a customer at  $90^{\circ}\text{C}$ . Two minutes later, the temperature of the beverage is  $80^{\circ}\text{C}$ . The room temperature is  $25^{\circ}\text{C}$ . After how many minutes is it at  $60^{\circ}\text{C}$ , which is the temperature preferred by the customer?
- 15) A frozen pizza at  $32^{\circ}\text{F}$  is removed from the freezer and is allowed to stand in a  $70^{\circ}\text{F}$  room. After 5 minutes the temperature of the pizza is  $40^{\circ}\text{F}$ . When will the pizza be  $60^{\circ}\text{F}$  ?



**Answer Key**

- 1)  $7500e^2$
- 2) 80,922 years
- 3) 1367
- 4) 4.630 pounds
- 5) .6635 grams or 663.6 milligrams
- 6) .069
- 7) a) 1020  
b) 230.259 hrs.
- 8) 866 years
- 9) a) 25.379  
b) 8.844
- 10) a) 8.267  
b) 3.553
- 11) a) 17.527 more min.  
b) 13.258
- 12) -2.562 so person was murdered around 5:30
- 13) 20.830 min more
- 14) 7.411
- 15) 28.237 min

