Unit #8: Taylor Polynomials and Power Series
Topic: Lagrange Error Bound
Objective: SWBAT find the error in a Taylor approximation by using the Lagrange error bound formula.

Warm Up #7:

The *nth* derivative of a function f at $x = 0$ is given by $f^{(n)}(0) = (-1)^n \frac{n+1}{(n+2)2^n}$ for all $n \geq 0$. Which of the following is the Maclaurin series for f?

(a) $-\frac{1}{2} + \frac{1}{3}x - \frac{3}{32}x^2 + \frac{1}{60}x^3 - \cdots$

(b) $\frac{1}{2} - \frac{1}{3}x + \frac{3}{16}x^2 - \frac{1}{10}x^3 + \cdots$

(c) $\frac{1}{2} + \frac{1}{3}x + \frac{3}{32}x^2 + \frac{1}{60}x^3 + \cdots$

(d) $\frac{1}{2} - \frac{1}{3}x + \frac{3}{32}x^2 - \frac{1}{60}x^3 + \cdots$

(e) $\frac{1}{2} - 3x + \frac{32}{3}x^2 - 60x^3 + \cdots$

Lagrange Error Bound

When a Taylor polynomial is used to approximate a function, we need a way to see how accurate the approximation is. In other words, how large the remainder (error) may be.

$\boldsymbol{f(x)}$	$=$	$\boldsymbol{P_n(x)}$	$+$	$\boldsymbol{R_n(x)}$	*so*	$\boldsymbol{R_n(x)}$	$=$	$\boldsymbol{f(x)}$	$-$	$\boldsymbol{P_n(x)}$
Function	=	Polynomial Approximation	+	Remainder		Remainder	=	Function	-	Polynomial Approximation

<u>Taylor's Theorem:</u>

If a function f is differentiable through order $n+1$ in an interval containing c, then for each x in the interval, there exists a number a between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^n(c)}{n!}(x-c)^n + R_n(x)$$

where the maximum remainder $R_n(x)$ (or error) is given by

$$R_n(x) < max\left|\frac{f^{n+1}(a)(x-c)^{n+1}}{(n+1)!}\right|$$

also known as ***The Lagrange Remainder (or Error Bound).***

When applying Taylor's Formula, we cannot find the exact value of a. Rather, we are merely interested in a safe upper bound from which we will be able to tell how large the remainder is.

Lagrange Error Bound

Example 1:
Calculator Permitted

Let f be a function with 5 derivatives on the interval $[2,3]$. Assume that $\left|f^{(5)}(x)\right| < 0.2$ for all x in the interval $[2,3]$ and that a fourth-degree Taylor polynomial for f at $c = 2$ is used to estimate $f(3)$

(a) How accurate is this approximation? Give three decimal places.

(b) Suppose that $P_4(3) = 1.763$. Use your answer to (a) to find an interval in which $f(3)$ must reside.

(c) Could $f(3)$ equal 1.778? Why or why not?

(d) Could $f(3)$ equal 1.764? Why or why not?

Example 2:
Calculator Permitted

(a) Find the fifth-degree Maclaurin polynomial for $\sin x$. Then use your polynomial to approximate $\sin 1$, and use Taylor's Theorem to find the maximum error for your approximation. Give three decimal places.

(b) Use your answer to (a) to find an interval $[a, b]$ such that $a \le \sin 1 \le b$.

(c) Could $\sin 1$ equal 0.9? Why or why not?

Example 3:
No Calculator

(a) Write the fourth-degree Maclaurin polynomial for $f(x) = e^x$. Then use your polynomial to approximate e, and find a Lagrange error bound for the maximum error when $|x| \le 1$.

(b) Use your answer to (a) to find an interval $[a,b]$ such that $a \le e \le b$.

Example 4:
Calculator Permitted

The function *f* has derivatives of all orders for all real numbers x. Assume that $f(2) = 6$, $f'(2) = 4$, $f''(2) = -7$, $f'''(2) = 8$.

(a) Write the third-degree Taylor polynomial for *f* about $x = 2$, and use it to approximate $f(2.3)$. Give three decimal places.

(b) The fourth derivative of *f* satisfies the inequality $\left|f^{(4)}(x)\right| \le 9$ for all x in the closed interval $[2, 2.3]$. Use the Lagrange error bound on the approximation of $f(2.3)$ found in part (a) to find an interval $[a,b]$ such that $a \le f(2.3) \le b$. Give three decimal places.

(c) Based on the information above, could $f(2.3)$ equal 6.992? Explain why or why not.

Unit 9 – **Lagrange Error Bound**

Show all work. Calculator permitted except unless specifically stated.

Free Response & Short Answer

5) (a) Find the fourth-degree Taylor polynomial for $\cos x$ about $x = 0$. Then use your polynomial to approximate the value of $\cos 0.8$, and use Taylor's Theorem to determine the accuracy of the approximation. Give three decimal places.

(b) Find the interval $[a, b]$ such that $a \le \cos 0.8 \le b$.

(c) Could $\cos 0.8$ equal 0.695? Show why or why not.

Lagrange Error Bound

6) (a) Write a fourth-degree Maclaurin polynomial for $f(x) = e^x$. Then use your polynomial to approximate e^{-1}, and find a Lagrange error bound for the maximum error when $|x| \leq 1$. Give three decimal places.

(b) Find an interval $[a, b]$ such that $a \leq e^{-1} \leq b$.

Lagrange Error Bound

7) Let f be a function that has derivatives of all orders for all real numbers x. Assume that $f(5)=6$, $f'(5)=8$, $f''(5)=30$, $f'''(5)=48$, and $\left|f^{(4)}(x)\right| \le 75$ for all x in the interval $[5,5.2]$.

(a) Find the third-degree Taylor polynomial about $x=5$ for $f(x)$.

(b) Use your answer to part (a) to estimate the value of $f(5.2)$. What is the maximum possible error in making this estimate? Give three decimal places.

(c) Find an interval $[a,b]$ such that $a \le f(5.2) \le b$. Give three decimal places.

(d) Could $f(5.2)$ equal 8.254? Show why or why not.

Lagrange Error Bound

8) Let f be a function that has derivatives of all orders on the interval $(-1,1)$. Assume $f(0)=1$, $f'(0)=\frac{1}{2}$, $f''(0)=-\frac{1}{4}$, $f'''(0)=\frac{3}{8}$, and $\left|f^{(4)}(x)\right|\leq 6$ for all x in the interval $(-1,1)$.

(a) Find the third-degree Taylor polynomial about $x=0$ for the function f.

(b) Use your answer to part (a) to estimate the value of $f(0.5)$.

(c) What is the maximum possible error for the approximation made in part (b)?

Answer Key:

1) a) .002	b) $1.761 \leq f(3) \leq 1.765$	c) no, not in interval
d) yes	2) a) .001	b) $.841 \leq sin1 \leq .843$
c) no, not in interval	3) a) $e = \frac{65}{24}$, $error < \frac{3}{120}$	b) $2.683 \leq e \leq 2.733$
4) a) 6.921	b) $error < .003$, $6.918 \leq f(2.3) \leq 6.924$	
c) no, not in interval	5) a) $cos.8 \approx .697$, $error < .003$	b) $.694 \leq cos.8 \leq .700$
c) yes	6) a) $e^{-1} \approx .375$, $error < .023$	b) $.352 \leq e^{-1} \leq .398$
7) a) $T_3(x) = 6 + 8(x-5) + 15(x-5)^2 + 8(x-5)^3$		b) $f(5.2) \approx 8.264$ $error < .005$
c) $8.259 \leq f(5.2) \leq 8.269$		d) no, not in interval
8) a) $T_3(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$		b) 1.227
c) $error < .0156$		