Unit #3: Differential Equations
Topic: Logistic Growth
Objective: SWBAT solve logistic growth problems by using differential equations.

Warm Up #9:

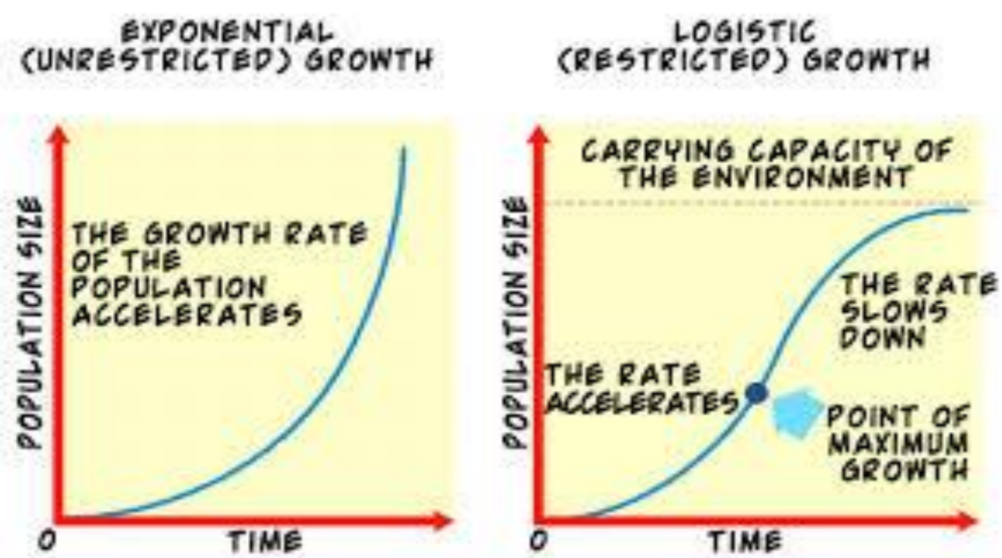
Solve the initial value problem: $\frac{dy}{dx} = \frac{4\sqrt{y}\,lnx}{x}$ when $y = 1$ and $x = e$.

We have already seen the exponential growth model given by $y = C_0e^{kt}$.

However, in real-life things do not always increase forever. For example, if we consider a population model, many times there is a limiting factor such as food, living space, etc.

There is a maximum population, or **carrying capacity**, L.

Logistic growth occurs when the growth rate slows as the population approaches a maximal sustainable population L.



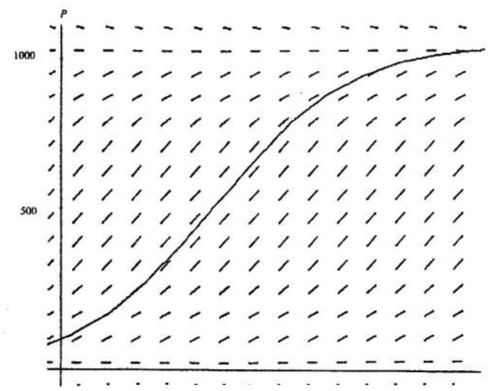
The Logistic Growth Model

The **logistic growth model** is where growth rate is proportional to **BOTH** the amount present (y) and the carrying capacity that remains: $(L - y)$

Example:

Due to limited food and space, a squirrel population cannot exceed 1000. It grows at a rate proportional both to the existing population and to the attainable additional population. If there were 100 squirrels 2 years ago, and 1 year ago the population was 400, about how many squirrels are there now?

The following shows the slope field for the equation given above. Notice the behavior or the slopes. Where are the slopes the steepest?



Problem Set #9:

1) Suppose a flu-like virus is spreading through a population of 50,000 at a rate proportional both to the number of people already infected and to the number still unaffected. If 100 people were infected yesterday and 130 are infected today.

(a) Write an expression for the number of people $N(t)$ infected after t days.
(b) Determine how many will be infected a week from today.
(c) Indicate when the virus will be spreading the fastest.

2) The number of people that hear a rumor follows logistic growth. In a school of 1500 students, 5 students start a rumor. After 2 hours, 120 students have heard about the rumor. Find the number of students to hear the rumor after one more hour.

3) A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right)$, $40 \le p \le 4000$ where t is the number of years.

(a) Write a model for the elk population in terms of t.
(b) Use the model to estimate the elk population after 15 years.
(c) Find the limit of the model as $t \to \infty$.

4) A certain population has 10,000 people. A disease is spreading through the population at a rate proportional to both the infected population and to the unaffected population. If it is known that 1,000 were infected two months ago, and 4,000 were infected last month, how many should be infected this month? How many months from now will 90% of the population be infected?

5) Suppose a rumor is spreading at a dance attended by 200 students. The rumor is spreading at a rate that is directly proportional to both the number of students who have heard the rumor and the number of students who have not heard the rumor. Let P be the number of students who have heard the rumor, and let t be the time in minutes since the rumor began to spread.

(a) Write a differential equation to model this rate of change.
(b) If $P(0) = 10$ and $P(15) = 50$, solve for P as a function of t.
(c) Use your solution to (b) to find the number of students who have heard the rumor after 1 hour.
(d) Use your solution to (b) to find the time it takes for 175 students to hear the rumor.

6) Ten grizzly bears were introduced to a national park 10 years ago. There are 23 bears in the park at the present time. The park can support a maximum of 100 bears. Assuming a logistic growth model, when will the bear population reach 75?

7) Two dogs infected with a virus are admitted to a veterinary clinic, which holds a maximum of fifty dogs. The virus spread among the animals, and two days later ten dogs were infected. If spread of the virus is not halted, after how many days will 25 dogs be infected?

8) The rate of growth of the fish population in an artificial pond is given by the differential equation $\frac{dy}{dt} = 0.003y(500 - y)$, where y is the number of fish in the pond after t days.

(a) What is the maximum number of fish that the fish pond can hold?
(b) Write the solution y to the differential equation if the pond is populated initially with 50 fish.
(c) What is the $y - value$ of the point of inflection?
(d) Find the number of fish in the pond after 3 days.

Answer Key

1. a) $y = \frac{50{,}000}{1+499e^{ln(4987/6487)t}}$ b) 808 people   c) when 25,000 people are infected

2. 461 students

3. a) $y = \frac{4{,}000}{1+Ce^{-4000kt}}$ b) 629   c) 4000

4. 8000 people; .453 months from now

5. a) $\frac{dy}{dt} = ky(200 - y)$ b) $y = \frac{200}{1+19e^{\left(\frac{ln\left(\frac{3}{19}\right)}{15}\right)t}}$ c) 198   d) 39.741 min

6. 33.328 years

7. 3.547 days

8. a) 500   b) $y = \frac{500}{1+9e^{-1.5t}}$ c) 250   d) 454.553