Unit: Taylor Polynomials and Power Series
Topic: Taylor Polynomial and Power Series Review
Objective: SWBAT repair the skills needed to answer questions on Taylor Polynomials and Power Series for the upcoming exam.

Directions: Read each question carefully and show all work.

1) Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5$, $f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$.

   (a) Write the third–degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.

   (b) Write the fourth–degree Taylor polynomial for g, where $g(x) = f(x^2)$, about $x = 0$.

   (c) Write the third–degree Taylor polynomial for h, where $h(x) = \int_0^x f(t)\,dt$, about $x = 0$.

2) Suppose the function $f(x)$ is approximated near $x = 0$ by a sixth-degree Taylor polynomial $P_6(x) = 3x - 4x^3 + 5x^6$. Give the value of each of the following:

   a) $f(0)$ b) $f'(0)$ c) $f'''(0)$

   d) $f^5(0)$ e) $f^6(0)$

3) The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third–degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

(b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.

(c) Write the fourth–degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

4) Let $f(x) = cos(2x) - 1 + 2x^2$.

a) Find the first two non-zero terms in the Maclaurin series expansion of f.

b) Using the expansion found in part (a) compute: $\lim_{x \to 0} \frac{cos(2x)-1+2x^2}{x^4}$

5) The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

on its interval of convergence.

a) Find the interval of convergence of the Maclaurin series for f. Justify your answer.
b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.
c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

6) Find the first four nonzero terms and the general term for the Macluarin series for $\int_0^x cost^2 dt$.

7) a) Write the Taylor series expansion about $x = 0$ for $f(x) = ln(1 + x^2)$. Include an expression for the general term.

b) For what values of x does the series in part (a) converge?

c) Estimate the error in evaluating $ln\left(\frac{13}{9}\right)$ by using only the first four nonzero terms of the series in part (a). Justify your answer.

8) a) Suppose a function f is approximated with a fifth-degree Taylor polynomial about $x = 2$. If the maximum value of the sixth derivative between $x = 2$ and $x = 5$ is 0.25, that is $\left|f^{(6)}(x)\right| < 0.25$, then find the maximum error incurred using the approximation to compute f(5).

b) Suppose $P_5(5) = 3.618$. Use your answer to (a) to find an interval in which $f(5)$ must reside.

c) Could $f(5) = 3.725$? Why or why not?

9) The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \geq 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

(b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

(d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

10) Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, where $a_0 = 1$ and $a_n = \left(\frac{7}{n}\right) a_{n-1}$ for $n \geq 1$.

(a) Find the first four terms and the general term of the series.
(b) For what values of x does the series converge?
(c) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, find the value of $f'(1)$.

Answer Key

1) a) 4.425	b) $P_4(x) = 5 - 3x^2 + \frac{1}{2}x^4$	c) $P_3(x) = 5x - \frac{3}{2}x^2 + \frac{1}{6}x^3$
2) a) 0 b) 3 c) -24 d) 0 e) 3600		3) a) $-\frac{119}{24}$
b) $error \le \frac{1}{128}$, $-4.966 < f(1.5) < -4.951$	c) $P(x) = -3 + 5x^2 + \frac{3}{2}x^4$	
4) a) $f(x) = \frac{2}{3}x^4 - \frac{4}{45}x^6$	b) 2/3	5) a) $-\frac{1}{2} \le x < \frac{1}{2}$
b) $f'(x) = 2 + 4x + 8x^2 + 16x^3 + \cdots 2(2x)^n + \cdots$	c) 6/5	
6) $x - \frac{x^5}{5(2!)} + \frac{x^9}{9(4!)} - \frac{x^{13}}{13(6!)} + \cdots \frac{(x^2)^{2n+1}}{(2n+1)!} + \cdots$		
7) a) $\ln(1 + x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots \frac{(-1)^{n+1}x^{2n}}{n} + \cdots$	b) $-1 \le x \le 1$ c) .00000006	
8) a) .253125 b) $3.365 \le f(5) \le 3.871$ c) yes		
9) a) $f(x) = 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots \frac{n+1}{3^n}(x-2)^n + \cdots$		
b) $R = 3$	c) $g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \cdots \frac{1}{3^n}(x-2)^{n+1} + \cdots$	
d) no, interval of convergence is $-1 < x < 5$		
10) a) $1 + 7x + \frac{7^2x^2}{2!} + \frac{7^3x^3}{3!} + \cdots \frac{(7x)^n}{n!} + \cdots$		
b) converges for all real numbers	c) $f'(1) = 7e^7$	